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TECHNICAL REPORT: COMMUNICATIONS

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PART 2

THE DIFFRACTION PATTERN IN THE PRESENCE OF SINGLE THIRD-ORDER ABERRATIONS:

1. THE EFFECT OF PRIMARY ASTIGMATISM

NOTICE

WORK CARRIED OUT AS PART OF THE LOCKHEED INDEPENDENT RESEARCH PROGRAM.

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THE DIFFRACTION PATTERN IN THE PRESENCE OF SINGLE THIRD-ORDER ABERRATIONS: 1. THE EFFECT OF PRIMARY ASTIGMATISM

by
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WORK CARRIED OUT AS PART OF THE LOCKHEED INDEPENDENT RESEARCH PROGRAM

Lockheed

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FOREWORD

This paper was prepared at the Electronics Sciences Laboratory as part of the Independent Research program of LMSC. It is the second of a series of contributions toward finding appropriate diffraction pattern manipulation techniques to improve the discrimination capability of aberrant optical systems for off-axis objects and deals with the diffraction in astigmatic systems. Further reports will be concerned with other types of third-order aberrations.

ABSTRACT

On the basis of the general solution of the diffraction integral for a rectangular aperture in the presence of geometrical-optical aberrations, obtained in Part I of this series, this paper deals with the effect of primary astigmatism upon the diffraction-pattern intensity distribution.

The diffraction pattern of a stationary rectangular aperture is described by means of a conveniently chosen Cartesian reference system. We consider two orthogonal aperture orientations and show that the intensity distribution depends on the orientation of the aperture's symmetry axes with respect to the meridional plane defined by the optical axis and the geometrical image point. The intensity distributions in special cases of interest are indicated for each of the two aperture orientations.

In the case of a slit rotating about the center of the entrance-pupil plane, i.e., in the case of arbitrary slit orientation with respect to the meridional plane, the Cartesian reference system is one which rotates synchronously with the aperture. We derive a general expression for the diffraction-pattern intensity distribution in this rotating coordinate system, and obtain its mean value throughout half a slit revolution by integration. Again we indicate special cases of interest.

Finally, we determine the overall diffraction-pattern intensity distribution of the rotating aperture with respect to a space-fixed coordinate system. The accumulated intensity at any arbitrary point of a space-fixed observation plane, as experienced for example, by a photographic plate, may be obtained by integration. We expect that the accumulated intensity at the Gaussian image point due to the continuously rotating slit has a greater predominance over the accumulated intensity at a point in the surrounding field than has the intensity at the geometrical image point over that at the field point when the rectangular aperture is stationary.

CONTENTS

LIST OF SYMBOLS

X, Y, Z	general coordinates; refer to Cartesian reference systems
X_1^*, Y_1^*	distances of the geometrical image point from the optical axis
R_1^*, ϕ	polar coordinates of the Gaussian image point in a rotated reference system
ξ, η	dimensionless coordinates of an aperture point or a point on the Gaussian reference sphere centered at the geometrical image point and passed through the center of the exit pupil
μ_o, ν_o	dimensionless parameters representing width and length of the rectangular aperture respectively
u, v, w	dimensionless optical coordinates of an observation point in the neighborhood of the Gaussian image point
Φ^ℓ	ℓ^{th} power of aberration function Φ
$K^\ell_{n-m,m}$	coefficient in power series representation of Φ^ℓ, n, m being running indices, ℓ a constant index
C	Seidel's aberration coefficient of primary astigmatism
$U(u, v, w)$	value of the disturbance at the observation point (u, v, w)
A	light-wave amplitude, assumed to be substantially constant over the wave-front portion which approximately fills the aperture.
λ	wavelength of monochromatic light

$k = \frac{2\pi}{\lambda}$	wave number
$V_\ell^{n-m}(u,v) = (-j)^{n-m}\,\overline{V_\ell^{n-m}(u,v)}$ $W_\ell^{m}(u,w) = (-j)^{m}\,\overline{W_\ell^{m}(u,w)}$	complex solution functions occurring in the expanded form of the disturbance $U(u, v, w)$; ℓ , n , m being running indices
$J_{r+1/2}(\xi)$	$(r + 1/2)^{th}$ order Bessel function of argument ξ , r being an integer
$G_r^k(z)$	solution function of argument z , occurring as factor in Bessel function series expansion of V_ℓ^{n-m} (or W_ℓ^m), r , k being running indices, defined on page 2-2
$N_{r,\rho}^k$, D_r^k	constant nominator and (common) denominator of coefficients occurring in Bessel function series expansion of $G_r^k(z)$, derived in Refs. 1 and 2, ρ being running index
$I(u, v, w)$	intensity at observation point (u, v, w)
$I_o(0, 0, 0) = \left(\frac{A}{\lambda} 4\mu_o\nu_o\right)^2$	intensity of the geometrical image point
$i(u, v, w) = I(u, v, w)/I_o$	normalized intensity at the observation point
$\overline{i(u, v, w)} = \frac{1}{\tau}\int_0^\pi i(u, v, w)\, d\phi$	mean intensity at relating point (u, v, w)
$i(u, v_o, w_o)$	instantaneous normalized intensity at space-fixed observation point (u, v_o, w_o) due to rotating aperture
ϕ_o	angle of initial orientation of the rectangular aperture with respect to the meridional plane
ϕ	angle of the slit rotation about the optical axis
ρ_o	radial distance of observation point from the u-axis

$i(u, v_o, w_o)_{tot} = \int_0^{\pi} i(u, v_o, w_o)\, d\phi$ **accumulated intensity at space-fixed observation point (u, v_o, w_o) due to the continuously rotating slit**

Section 1
INTRODUCTION

In a previous paper (Ref. 1) a general solution of Fresnel-Kirchhoff's diffraction integral was derived. This integral pertains to a rectangular aperture and represents the disturbance, due to diffraction, at a point in the neighborhood of the geometrical image of an object point in the presence of geometrical aberrations. However, the general expressions obtained for the disturbance and for the (three-dimensional) intensity distribution do not readily yield a precise insight into the combined effects of diffraction and aberrations.

For such an insight, it is helpful to consider single aberrations affecting the structure of the diffraction pattern.* The most important single aberrations are contained in Seidel's third-order aberration function. This paper deals with the effect of primary astigmatism.

*According to the theory of geometrical optical aberrations, the describing function, in general, is given by

$$\Phi = \sum_{k=2}^{\infty} \Phi^{(2k)}$$

where $\Phi^{(2k)}$ is a polynomial of order $(2k)$ in four significant variables. Every group of order $(2k - 1)$ splits into a finite summation of specific kinds of single aberrations, and each aberration has a different effect on image quality.

Section 2
THE DIFFRACTION INTEGRAL SOLUTION IN THE CASE OF ASTIGMATIC ABERRATIONS

If the meridional plane defined by the optical axis and the principal ray of the image-forming optical system is chosen as the YZ-plane of the Cartesian coordinate system used for describing the diffration pattern, the aberration function for primary astigmatism is represented in a very simple form.

In the notations of the previous paper (Ref. 1), for the ℓ^{th} power of the wave aberration, we have

$$\Phi^{\ell} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} K^{\ell}_{n-m,m} \; \xi^{n-m} \; \eta^{m}$$

where the parameters $K^{\ell}_{n-m,m}$ generally depend on

- The off-axis position of the geometrical image point
- The linear dimensions of the aperture
- The radius of the Gaussian reference sphere
- The lateral magnification of the pupil planes
- The aberration coefficients of the optical system

For the case of primary astigmatism, we have specifically

$$\Phi^{(4)} = - CY_1^{*2} \nu_o^2 \eta^2$$

Therefore, the coefficients are given by

$$K^{\ell}_{n-m,m} \equiv 0 \quad \text{for } (n - m) \neq 0 \text{ , } m \neq 2\ell$$

$$K^{\ell}_{o,2\ell} = (-1)^{\ell} \left(CY_1^{*2} \nu_o^2 \right)^{\ell}$$

The disturbance at the abservation point, with the "optical coordinates" u, v, w, takes the form

$$U(u, v, w) = - je^{ju} \frac{A}{\lambda} 4\mu_o\nu_o \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (jk)^{\ell} K^{\ell}_{o,2\ell} V^{o}_{\ell}(u, v) W^{2\ell}_{\ell}(u, w)$$

where

$$V^{o}_{\ell}(u, v) = \frac{\pi}{2} \sum_{s=0}^{\infty} (-j)^{s} (2s + 1) \frac{J_{s+1/2}\left(\frac{u}{2} \mu_o^2\right)}{\left(\frac{u}{2} \mu_o^2\right)^{1/2}} G^{o}_{s}(v\mu_o)$$

$$W^{2\ell}_{\ell}(u, w) = (-1)^{\ell} \frac{\pi}{2} \sum_{t=0}^{\infty} (-j)^{t} (2t + 1) \frac{J_{t+1/2}\left(\frac{u}{2} \nu_o^2\right)}{\left(\frac{u}{2} \nu_o^2\right)^{1/2}} G^{2\ell}_{t}(w\nu_o)$$

$$G^{o}_{s}(v\mu_o) = \sum_{\sigma=0}^{s} (-1)^{s-\sigma} \frac{N^{o}_{s,\sigma}}{D^{o}_{s}} \frac{J_{2s-2\sigma+1/2}(v\mu_o)}{(v\mu_o)^{1/2}}$$

$$G^{2\ell}_{t}(w\nu_o) = \sum_{\tau=0}^{t+\ell} (-1)^{t-\tau} \frac{N^{2\ell}_{t,\tau}}{D^{2\ell}_{t}} \frac{J_{2t+2\ell-2\tau+1/2}(w\nu_o)}{(w\nu_o)^{1/2}} \quad *$$

*The coefficients $N^{k}_{r,i}$ and D^{k}_{r} are available on request (Ref. 2)

The intensity distribution is given by

$$I(u, v, w) = I_o \left| \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (jk)^{\ell} K^{\ell}_{o, 2\ell} V^{o}_{\ell}(u, v) W^{2\ell}_{\ell}(u, w) \right|^2$$

where

$$I_o = \left(\frac{A}{\lambda} 4\mu_o \nu_o \right)^2$$

If the rectangular aperture is rotated by $\pi/2$, the two slit parameters μ_o and ν_o are interchanged.

By performing this interchangement, we see that for both cases the disturbance at a specific observation point (u, v, w), as well as the intensity distribution,are completely different.

The values of the disturbance function and the intensity distribution in three characteristic orthogonal planes are believed to be of special interest. These planes are

- The Gaussian image plane, defined by $u = 0$
- The meridional plane, defined by $v = 0$
- The tangential plane, which is perpendicular to the meridional plane and is defined by $w = 0$

Since we have

$$\lim_{\zeta \to o} \frac{J_{r+1/2}(\zeta)}{\zeta^{1/2}} = \sqrt{\frac{2}{\pi}} \, \delta^o_r$$

where δ_r^o is the Kronecker symbol, we obtain for $u = 0$ nonzero contributions only from terms for which $s = \sigma = t \equiv 0$. For the meridional plane orientation of the slit's longitudinal axis, this yields,

$$V_\ell^o(0, v) = \left(\frac{\sin v\mu_o}{v\mu_o}\right)$$

$$W_\ell^{2\ell}(0, w) = (-1)^\ell \sqrt{\frac{\pi}{2}} \sum_{\tau=0}^{\ell} (-1)^\tau \frac{N_{o,\tau}^{2\ell}}{D_o^{2\ell}} \frac{J_{2\ell-2\tau+1/2}(w\nu_o)}{(w\nu_o)^{1/2}}$$

$$i(0,v,w) = I(0,v,w)/I_o$$

$$= \left| \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (jk)^\ell \left(CY_1^{*2} \nu_o^2\right)^\ell \frac{\sin v\mu_o}{v\mu_o} \sqrt{\frac{\pi}{2}} \sum_{\tau=0}^{\ell} (-1)^\tau \frac{N_{o,\tau}^{2\ell}}{D_o^{2\ell}} \frac{J_{2\ell-2\tau+1/2}(w\nu_o)}{(w\nu_o)^{1/2}} \right|^2$$

$$i(0, v, 0) = \left| \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (-jk)^\ell \left(CY_1^{*2} \nu_o^2\right)^\ell \frac{N_{o,\ell}^{2\ell}}{D_o^{2\ell}} \frac{\sin v\mu_o}{v\mu_o} \right|^2$$

$$i(0, 0, w) = \left| \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (jk)^\ell \left(CY_1^{*2} \nu_o^2\right)^\ell \sqrt{\frac{\pi}{2}} \sum_{\tau=0}^{\ell} (-1)^\tau \frac{N_{o,\tau}^{2\ell}}{D_o^{2\ell}} \frac{J_{2\ell-2\tau+1/2}(w\nu_o)}{(w\nu_o)^{1/2}} \right|^2$$

$$i(0, 0, 0) = \left| \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (-jk)^\ell \left(CY_1^{*2} \nu_o^2\right)^\ell \frac{N_{o,\ell}^{2\ell}}{D_o^{2\ell}} \right|^2$$

If we replace ν_o by μ_o in the above expressions, we obtain the corresponding results for an aperture rotated by $\pi/2$.

We see that the intensity at the center of the diffraction pattern is not equal to unity, and, therefore, differs from the normalized central intensity of the aberration-free diffraction pattern. Furthermore, from interchanging μ_o and ν_o, it is evident that all intensities displayed are dependent on the orientation of the rectangular aperture with respect to the meridional plane.

In a similar manner, for the meridional plane $v = 0$, one obtains

$$i(u,0,w) = \left| \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (-jk)^{\ell} \left(CY_1^{*2} \nu_o^2 \right)^{\ell} \sqrt{\frac{\pi}{2}} \sum_{s=0}^{\infty} (-j)^s (2s+1) \frac{N_{s,s}^{o}}{D_s^{o}} \frac{J_{s+1/2}\left(u\mu_o^2/2\right)}{\left(u\mu_o^2/2\right)^{1/2}} W_{\ell}^{2\ell}(u,w) \right|^2$$

$$i(u,0,0) = \left| \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (-jk)^{\ell} \left(CY_1^{*2} \nu_o^2 \right)^{\ell} \sqrt{\frac{\pi}{2}} \sum_{s=0}^{\infty} (-j)^s (2s+1) \frac{N_{s,s}^{o}}{D_s^{o}} \frac{J_{s+1/2}\left(u\mu_o^2/2\right)}{\left(u\mu_o^2/2\right)^{1/2}} \right.$$

$$\left. \times \sqrt{\frac{\pi}{2}} \sum_{t=0}^{\infty} (-j)^t (2t+1) \frac{N_{t,t+\ell}^{2\ell}}{D_t^{2\ell}} \frac{J_{t+1/2}\left(u\nu_o^2/2\right)}{\left(u\nu_o^2/2\right)^{1/2}} \right|^2$$

Finally, for the tangential plane, $w = 0$, one gets

$$i(u,v,0) = \left| \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (-jk)^{\ell} \left(CY_1^{*2} \nu_o^2 \right)^{\ell} \sqrt{\frac{\pi}{2}} \sum_{t=0}^{\infty} (-j)^t (2t+1) \frac{N_{t,t+\ell}^{2\ell}}{D_t^{2\ell}} \frac{J_{t+1/2}\left(u\nu_o^2/2\right)}{\left(u\nu_o^2/2\right)^{1/2}} V_{\ell}^{o}(u,v) \right|^2$$

In these expressions, the corresponding results for an aperture rotated by $\pi/2$ with respect to the meridional plane are again obtained by interchanging μ_o and ν_o.

Section 3
DIFFRACTION BY A ROTATING RECTANGULAR APERTURE

In the preceding section, we have noted that the intensity distribution in the diffraction pattern of a rectangular aperture in the presence of primary astigmatism depends on the orientation of the slit with respect to the meridional plane containing the geometrical image point and the optical axis. Of course, it is desirable to determine the shape of the diffraction pattern for any arbitrary orientation of the rectangular aperture. This determination is of special interest when we consider an aperture which rotates with constant angular speed about the center of the entrance-pupil plane.

The solution of Fresnel-Kirchhoff's diffraction integral is usually found by expressing analytically, in the integrand, the optical paths involved. The intensity distribution is described with respect to a Cartesian reference coordinate system. This reference system has its origin at the (space-fixed) geometrical image point and its (positive) Z-axis in direction of the light propagation. Although the orientations of the X- and Y-axes may be chosen arbitrarily, it is convenient to place them respectively in the orthogonal planes defined by the principal ray and lines drawn parallel to the aperture's lateral and longitudinal symmetry axes. Without restricting the generality, we may pass the slit's symmetry axes through the center of the entrance-pupil plane.

When the aperture is rotated by a certain angle ϕ about the optical axis, the diffraction pattern rotates synchronously. However, the meridional plane and, therefore, the distance of the geometrical image point from the optical axis remain unchanged. For this reason, the Gaussian image-point position with respect to the reference system is expressed in polar coordinates as follows:

$$X_1^* = R_1^* \sin \phi \quad , \quad Y_1^* = R_1^* \cos \phi$$

The aberration function must now be represented in its general form

$$\Phi = - C \left[X_1^* \mu_o \xi + Y_1^* \nu_o \eta \right]^2$$

$$= (-1)^1 \left(C R_1^{*2} \right)^1 \left[(\mu_o \sin \phi) \xi + (\nu_o \cos \phi) \eta \right]^2$$

and its ℓ^{th} power by

$$\Phi^\ell = (-1)^\ell \left(C R_1^{*2} \right)^\ell \sum_{\lambda=0}^{2\ell} \binom{2\ell}{\lambda} (\mu_o \sin \phi)^{2\ell-\lambda} (\nu_o \cos \phi)^\lambda \xi^{2\ell-\lambda} \eta^\lambda$$

Hence

$$K^\ell_{2\ell-\lambda,\lambda} = (-1)^\ell \left(C R_1^{*2} \right)^\ell \binom{2\ell}{\lambda} (\mu_o \sin \phi)^{2\ell-\lambda} (\nu_o \cos \phi)^\lambda$$

This yields

$$U(u,v,w) = - je^{ju} \left(\frac{A}{\lambda} 4\mu_o \nu_o \right) \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (jk)^\ell \sum_{\lambda=0}^{2\ell} K^\ell_{2\ell-\lambda,\lambda} V_\ell^{2\ell-\lambda} (u,v) W_\ell^\lambda (u,w)$$

where

$$V_\ell^{2\ell-\lambda}(u,v) = (-j)^{2\ell-\lambda} \frac{\pi}{2} \sum_{s=0}^{\infty} (-j)^s (2s+1) \frac{J_{s+1/2}\left(\frac{u}{2} \mu_o^2\right)}{\left(\frac{u}{2} \mu_o^2\right)^{1/2}} G_s^{2\ell-\lambda} (v\mu_o)$$

$$W_\ell^\lambda(u,w) = (-j)^\lambda \frac{\pi}{2} \sum_{t=0}^{\infty} (-j)^t (2t+1) \frac{J_{t+1/2}\left(u\nu_o^2/2\right)}{(u\nu_o^2/2)^{1/2}} G_t^\lambda(w\nu_o)$$

$$G_s^{2\ell-\lambda}(v\mu_o) = \sum_{\sigma=0}^{\frac{2s+2\ell-\lambda-(1)}{2}} (-1)^{s-\sigma} \frac{N_{s,\sigma}^{2\ell-\lambda}}{D_s^{2\ell-\lambda}} \frac{J_{2s+2\ell-\lambda-2\sigma+1/2}(v\mu_o)}{(v\mu_o)^{1/2}}$$

$$G_t^\lambda(w\nu_o) = \sum_{\tau=0}^{\frac{2t+\lambda-(1)}{2}} (-1)^{t-\tau} \frac{N_{t,\tau}^\lambda}{D_t^\lambda} \frac{J_{2t+\lambda-2\tau+1/2}(w\nu_o)}{(w\nu_o)^{1/2}}$$

The (normalized) expression for the intensity distribution takes the form

$$i(u,v,w) = I(u,v,w)/I_o$$

$$= \left| \sum_{\ell=0}^{\infty} \frac{1}{\ell!}\left(-jkCR_1^{*2}\right)^\ell \sum_{\lambda=0}^{2\ell} \binom{2\ell}{\lambda} (\mu_o \sin\phi)^{2\ell-\lambda} (\nu_o \cos\phi)^\lambda V_\ell^{2\ell-\lambda}(u,v)\, W_\ell^\lambda(u,w) \right|^2$$

In accordance with the derivations of the preceding section, for $\phi = 0$, only the terms containing $\nu_o^{2\ell}$, and for $\phi = \pi/2$, only the terms containing $\mu_o^{2\ell}$ are nonzero. At every observation point (u,v,w), an oscillatory intensity is observed as ϕ varies between 0 and 2π.

Note that, in this derivation, the specific observation point chosen does not remain space-fixed, but travels along a circle about the u-axis in step with the aperture rotation.

Of special interest, again, are the intensity distributions

- In the geometrical image plane, $u = 0$
- In the plane $v = 0$
- In the plane $w = 0$
- Along the u-axis, $v = w = 0$
- Along the v-axis, $u = w = 0$
- Along the w-axis, $u = v = 0$
- At the diffraction pattern center in the Gaussian image plane, $u = v = w = 0$

In the plane $u = 0$, we obtain from

$$G_o^{2\ell-\lambda}(v\mu_o) = \sum_{\sigma=0}^{\frac{2\ell-\lambda-(1)}{2}} (-1)^\sigma \frac{N_{o,\sigma}^{2\ell-\lambda}}{D_o^{2\ell-\lambda}} \frac{J_{2\ell-\lambda-2\sigma+1/2}(v\mu_o)}{(v\mu_o)^{1/2}}$$

$$G_o^{\lambda}(w\nu_o) = \sum_{\tau=0}^{\frac{\lambda-(1)}{2}} (-1)^\tau \frac{N_{o,\tau}^{\lambda}}{D_o^{\lambda}} \frac{J_{\lambda-2\tau+1/2}(w\nu_o)}{(w\nu_o)^{1/2}}$$

$$V_\ell^{2\ell-\lambda}(0,v) = (-j)^{2\ell-\lambda}\sqrt{\frac{\pi}{2}} \sum_{\sigma=0}^{\frac{2\ell-\lambda-(1)}{2}} (-1)^\sigma \frac{N_{o,\sigma}^{2\ell-\lambda}}{D_o^{2\ell-\lambda}} \frac{J_{2\ell-\lambda-2\sigma+1/2}(v\mu_o)}{(v\mu_o)^{1/2}} = (-j)^{2\ell-\lambda}\,\overline{V_\ell^{2\ell-\lambda}(0,v)}$$

$$W_\ell^{\lambda}(0,w) = (-j)^{\lambda}\sqrt{\frac{\pi}{2}} \sum_{\tau=0}^{\frac{\lambda-(1)}{2}} (-1)^\tau \frac{N_{o,\tau}^{\lambda}}{D_o^{\lambda}} \frac{J_{\lambda-2\tau+1/2}(w\nu_o)}{(w\nu_o)^{1/2}} = (-j)^{\lambda}\,\overline{W_\ell^{\lambda}(0,w)}$$

$$V_\ell^{2\ell-\lambda}(0,v)\; W_\ell^{\lambda}(0,w) \;=\; (-1)^\ell\; \overline{V_\ell^{2\ell-\lambda}(0,v)}\; \overline{W_\ell^{\lambda}(0,w)}$$

the intensity distribution

i(0, v, w)

$$= \left| \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left(jkCR_1^{*2}\right)^\ell \sum_{\ell=0}^{2\ell} \binom{2\ell}{\lambda} \mu_o^{2\ell-\lambda}\, \nu_o^{\lambda}\, \overline{V_\ell^{2\ell-\lambda}(0,v)}\; \overline{W_\ell^{\lambda}(0,w)}\, \sin^{2\ell-\lambda}\phi\, \cos^{\lambda}\phi \right|^2$$

From this expression we obtain the distributions along the w- and v-axes. We see that in both cases only even values of ℓ are retained in the functions $G_o^{2\ell-\lambda}(0)$ and $G_o^{\lambda}(0)$ respectively. Defining these even values by $\lambda = 2\kappa$, one has $\sigma = \ell - \kappa$ along the w-axis, and $\tau = \kappa$ along the v-axis. This results in

$$G_o^{2\ell-2\kappa}(0) = (-1)^{\ell-\kappa} \sqrt{\frac{2}{\pi}}\, \frac{N_{o,\ell-\kappa}^{2\ell-2\kappa}}{D_o^{2\ell-2\kappa}}$$

$$G_o^{2\kappa}(0) = (-1)^{\kappa} \sqrt{\frac{2}{\pi}}\, \frac{N_{o,\kappa}^{2\kappa}}{D_o^{2\kappa}}$$

$$V_\ell^{2\ell-2\kappa}(0,0) = (-j)^{2\ell-2\kappa}\, (-1)^{\ell-\kappa}\, \frac{N_{o,\ell-\kappa}^{2\ell-2\kappa}}{D_o^{2\ell-2\kappa}}$$

$$W_\ell^{2\kappa}(0,0) = (-j)^{2\kappa}\, (-1)^{\kappa}\, \frac{N_{o,\kappa}^{2\kappa}}{D_o^{2\kappa}}$$

and yields the intensity distributions

$$i(0,0,w) = \Bigg| \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left(jkCR_1^{*2}\right)^{\ell} \sum_{\ell=0}^{\ell} \binom{2\ell}{2\kappa} \mu_o^{2\ell-2\kappa} \nu_o^{2\kappa} \sin^{2\ell-2\kappa}\phi \cos^{2\kappa}\phi$$

$$\times (-1)^{\ell-\kappa} \frac{N_{o,2-\kappa}^{2\ell-2\kappa}}{D_o^{2\ell-2\kappa}} \sqrt{\frac{\pi}{2}} \sum_{\tau=0}^{\kappa} (-1)^{\tau} \frac{N_{o,\tau}^{2\kappa}}{D_o^{2\kappa}} \frac{J_{2\kappa-2\tau+1/2}(w\nu_o)}{(w\nu_o)^{1/2}} \Bigg|^2$$

$$i(0,v,0) = \Bigg| \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left(jkCR_1^{*2}\right)^{\ell} \sum_{\kappa=0}^{\ell} \binom{2\ell}{2\kappa} \mu_o^{2\ell-2\kappa} \nu_o^{2\kappa} \sin^{2\ell-2\kappa}\phi \cos^{2\kappa}\phi$$

$$\times (-1)^{\kappa} \frac{N_{o,\kappa}^{2\kappa}}{D_o^{2\kappa}} \sqrt{\frac{\pi}{2}} \sum_{\sigma=0}^{\ell-\kappa} (-1)^{\sigma} \frac{N_{o,\sigma}^{2\ell-2\kappa}}{D_o^{2\ell-2\kappa}} \frac{J_{2\ell-2\kappa-2\sigma+1/2}(v\mu_o)}{(v\mu_o)^{1/2}} \Bigg|^2$$

Hence, at the geometrical image point, the intensity is given by

$i(0,0,0)$

$$= \Bigg| \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left(-jkCR_1^{*2}\right)^{\ell} \sum_{\kappa=0}^{\ell} \binom{2\ell}{2\kappa} \mu_o^{2\ell-2\kappa} \nu_o^{2\kappa} \sin^{2\ell-2\kappa}\phi \cos^{2\kappa}\phi \frac{N_{o,\ell-\kappa}^{2\ell-2\kappa}}{D_o^{2\ell-2\kappa}} \frac{N_{o,\kappa}^{2\kappa}}{D_o^{2\kappa}} \Bigg|^2$$

For the plane $v = 0$, from

$$G_s^{2\ell-2\kappa}(0) = (-1)^{\ell-\kappa} \sqrt{\frac{2}{\pi}} \frac{N_{s,s+\ell-\kappa}^{2\ell-2\kappa}}{D_s^{2\ell-2\kappa}}$$

$$V_\ell^{2\ell-2\kappa}(u,0) = (-j)^{2\ell-2\kappa}(-1)^{\ell-\kappa}\sqrt{\frac{\pi}{2}}\sum_{s=0}^{\infty}(-j)^s(2s+1)\frac{N_{s,s+\ell-\kappa}^{2\ell-2\kappa}}{D_s^{2\ell-2\kappa}}\frac{J_{s+1/2}\left(\frac{u}{2}\mu_o^2\right)}{\left(\frac{u}{2}\mu_o^2\right)^{1/2}}$$

we obtain the distribution

$$i(u,0,w) = \left|\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left(-jkCR_1^{*2}\right)^{\ell}\sum_{\kappa=0}^{\ell}\binom{2\ell}{2\kappa}\mu_o^{2\ell-2\kappa}\nu_o^{2\kappa}\sin^{2\ell-2\kappa}\phi\cos^{2\kappa}\phi\right.$$
$$\left.\times W_\ell^{2\kappa}(u,w)\sqrt{\frac{\pi}{2}}\sum_{s=0}^{\infty}(-j)^s(2s+1)\frac{N_{s,s+\ell-\kappa}^{2\ell-2\kappa}}{D_s^{2\ell-2\kappa}}\frac{J_{s+1/2}\left(\frac{u}{2}\mu_o^2\right)}{\left(\frac{u}{2}\mu_o^2\right)^{1/2}}\right|^2$$

where

$$W_\ell^{2\kappa}(u,w) = (-1)^{\kappa}\frac{\pi}{2}\sum_{t=0}^{\infty}(-j)^t(2t+1)\frac{J_{t+1/2}\left(\frac{u}{2}\nu_o^2\right)}{\left(\frac{u}{2}\nu_o^2\right)^{1/2}}G_t^{2\kappa}(w\nu_o)$$

$$G_t^{2\kappa}(w\nu_o) = \sum_{\tau=0}^{t+\kappa}(-1)^{t-\tau}\frac{N_{t,\tau}^{2\kappa}}{D_t^{2\kappa}}\frac{J_{2t+2\kappa-2\tau+1/2}(w\nu_o)}{(w\nu_o)^{1/2}}$$

Similarly, for the plane $w = 0$, one has

$$i(u,v,0) = \left|\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left(-jkCR_1^{*2}\right)^{\ell}\sum_{\kappa=0}^{\ell}\binom{2\ell}{2\kappa}\mu_o^{2\ell-2\kappa}\nu_o^{2\kappa}\sin^{2\ell-2\kappa}\phi\cos^{2\kappa}\phi\right.$$
$$\left.\times V_\ell^{2\ell-2\kappa}(u,v)\sqrt{\frac{\pi}{2}}\sum_{t=0}^{\infty}(-j)^t(2t+1)\frac{N_{t,t+\kappa}^{2\kappa}}{D_t^{2\kappa}}\frac{J_{t+1/2}\left(\frac{u}{2}\nu_o^2\right)}{\left(\frac{u}{2}\nu_o^2\right)^{1/2}}\right|^2$$

where

$$V_{\ell}^{2\ell-2\kappa}(u,v) = (-1)^{\ell-\kappa}\frac{\pi}{2}\sum_{s=0}^{\infty}(-j)^{s}(2s+1)\frac{J_{s+1/2}\left(\frac{u}{2}\mu_o^2\right)}{\left(\frac{u}{2}\mu_o^2\right)^{1/2}}G_s^{2\ell-2\kappa}(v\mu_o)$$

$$G_s^{2\ell-2\kappa}(v\mu_o) = \sum_{\sigma=0}^{s+\ell-\kappa}(-1)^{s-\sigma}\frac{N_{s,\sigma}^{2\ell-2\kappa}}{D_s^{2\ell-2\kappa}}\frac{J_{2s+2\ell-2\kappa-2\sigma+1/2}(v\mu_o)}{(v\mu_o)^{1/2}}$$

Along the u-axis, finally, the intensity distribution is determined by

$$i(u,0,0) = \left|\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left(-jkR_1^{*2}\right)^{\ell}\sum_{\kappa=0}^{\ell}\binom{2\ell}{2\kappa}\mu_o^{2\ell-2\kappa}\nu_o^{2\kappa}\sin^{2\ell-2\kappa}\phi\cos^{2\kappa}\phi\right.$$

$$\times\sqrt{\frac{\pi}{2}}\sum_{s=0}^{\infty}(-j)^{s}(2s+1)\frac{N_{s,s+\ell-\kappa}^{2\ell-2\kappa}}{D_s^{2\ell-2\kappa}}\frac{J_{s+1/2}\left(\frac{u}{2}\mu_o^2\right)}{\left(\frac{u}{2}\mu_o^2\right)^{1/2}}$$

$$\left.\times\sqrt{\frac{\pi}{2}}\sum_{t=0}^{\infty}(-j)^{t}(2t+1)\frac{N_{t,t+\kappa}^{2\kappa}}{D_t^{2\kappa}}\frac{J_{t+1/2}\left(\frac{u}{2}\nu_o^2\right)}{\left(\frac{u}{2}\nu_o^2\right)^{1/2}}\right|^2$$

As indicated on page 3-3, if the rectangular aperture is rotated about the optical axis, a specific point (u, v, w) of the observation plane travels along a circle about the u-axis. Because the intensity values encountered by this point change in an oscillatory manner, the determination of the mean intensity value which the traveling observation point experiences along its circular path should be of interest. The slit orientation is repeated at every half revolution; consequently we have, in usual notation,

$$\overline{i(u,v,w)} = \frac{1}{\pi}\int_0^{\pi} i(u,v,w)\,d\phi$$

For convenience, the evaluation of this integral may be restricted to the Gaussian image plane, since, for $u = 0$, the intensity-distribution function contains the imaginary unit only as a factor of the constant $\left(kCR_1^{*2}\right)$. Expressing $i(0,v,w)$ in its component form, one obtains

$$i(0,v,w) = \Bigg| \left\{ \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{1}{(2\ell)!} \left(kCR_1^{*2}\right)^{2\ell} \sum_{\lambda=0}^{4\ell} \binom{4\ell}{\lambda} \mu_o^{4\ell-\lambda} \nu_o^{\lambda} (\sin\phi)^{4\ell-\lambda} (\cos\phi)^{\ell} \right.$$

$$\left. \times \overline{V_{2\ell}^{4\ell-\lambda}(0,v)}\ \overline{W_{2\ell}^{\lambda}(0,w)} \right\}$$

$$+ j \left\{ \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{1}{(2\ell+1)!} \left(kCR_1^{*2}\right)^{2\ell+1} \sum_{\lambda=0}^{4\ell+2} \binom{4\ell+2}{\lambda} \mu_o^{4\ell+2-\lambda} \nu_o^{\lambda} (\sin\phi)^{4\ell+2-\lambda} (\cos\phi)^{\lambda} \right.$$

$$\left. \times \overline{V_{2\ell+1}^{4\ell+2-\lambda}(0,v)}\ \overline{W_{2\ell+1}^{\lambda}(0,w)} \right\} \Bigg|^2$$

In performing the integration, we encounter integrals of the form

$$\int_0^{\pi} (\sin\phi)^{4(\ell+\ell')-(\lambda+\lambda')} (\cos\phi)^{\lambda+\lambda'} d\phi$$

and

$$\int_0^{\pi} (\sin\phi)^{4(\ell+\ell')+4-(\lambda+\lambda')} (\cos\phi)^{\lambda+\lambda'} d\phi$$

respectively. These integrals are known to become zero if $(\lambda + \lambda')$ is odd. For even values of $(\lambda + \lambda')$, their solutions are given by

$$\pi \sum_{\rho=0}^{(\lambda+\lambda')/2} (-1)^{\rho} \binom{(\lambda+\lambda')/2}{\rho} \binom{4(\ell+\ell')-(\lambda+\lambda')+2\rho}{2(\ell+\ell')+(\lambda+\lambda')/2+\rho} \frac{1}{2^{4(\ell+\ell')-(\lambda+\lambda')+2\rho}}$$

and

$$\pi \sum_{\rho=0}^{(\lambda+\lambda')/2} (-1)^{\rho} \binom{(\lambda+\lambda')/2}{\rho} \binom{4(\ell+\ell')+4+(\lambda+\lambda')+2\rho}{2(\ell+\ell')+2+(\lambda+\lambda')/2+\rho} \frac{1}{2^{4(\ell+\ell')+4-(\lambda+\lambda')+2\rho}}$$

respectively.

Hence

$$\begin{aligned}
\overline{I(0,v,w)} = & \left\{ \sum_{\ell=0}^{\infty} \sum_{\ell'=0}^{\infty} (-1)^{\ell+\ell'} \frac{1}{(2\ell)!\,(2\ell')!} \left(kCR_1^{*2}\right)^{2(\ell+\ell')} \right. \\
& \times \sum_{\lambda=0}^{4\ell} \sum_{\lambda'=0}^{4\ell'} \binom{4\ell}{\lambda} \binom{4\ell'}{\lambda'} \mu_o^{4(\ell+\ell')-(\lambda+\lambda')} \nu_o^{\lambda+\lambda'} \overline{V_{2\ell}^{4\ell-\lambda}(0,v)}\; \overline{V_{2\ell}^{4\ell-\lambda}(0,v)}\; \overline{W_{2\ell}^{\lambda}(0,w)}\; \overline{W_{2\ell'}^{\lambda'}(0,w)} \\
& \left. \times \sum_{\rho=0}^{(\lambda+\lambda')/2} (-1)^{\rho} \binom{(\lambda+\lambda')/2}{\rho} \binom{4(\ell+\ell')-(\lambda+\lambda')+2\rho}{2(\ell+\ell')-(\lambda+\lambda')/2+\rho} \frac{1}{2^{4(\ell+\ell')-(\lambda+\lambda')+2\rho}} \right\} \\
& + \left\{ \sum_{\ell=0}^{\infty} \sum_{\ell'=0}^{\infty} (-1)^{\ell+\ell'} \frac{1}{(2\ell+1)!\,(2\ell'+1)!} \left(kCR_1^{*2}\right)^{2(\ell+\ell')+2} \right. \\
& \times \sum_{\lambda=0}^{4\ell+2} \sum_{\lambda'=0}^{4\ell'+2} \binom{4\ell+2}{\lambda} \binom{4\ell'+2}{\lambda'} \mu_o^{4(\ell+\ell')+4-(\lambda+\lambda')} \nu_o^{\lambda+\lambda'} \overline{V_{2\ell+1}^{4\ell+2-\lambda}(0,v)}\; \overline{V_{2\ell'+1}^{4\ell'+2-\lambda'}(0,v)}\; \overline{W_{2\ell+1}^{\lambda}(0,w)}\; \overline{W_{2\ell'+1}^{\lambda'}(0,w)} \\
& \left. \times \sum_{\rho=0}^{(\lambda+\ell')/2} (-1)^{\rho} \binom{(\lambda+\lambda')/2}{\rho} \binom{4(\ell+\ell')+4-(\lambda+\lambda')+2\rho}{2(\ell+\ell')+2-(\lambda+\lambda')/2+\rho} \frac{1}{2^{4(\ell+\ell')+4-(\lambda+\lambda')+2\rho}} \right\}
\end{aligned}$$

This formula gives a rough idea about the mean intensity encountered in the (rotating) diffraction pattern of the geometrical image plane of the continuously rotating rectangular aperture. In practice, however, this diffraction pattern will hardly be observable. Thus, the formula derived does not give precise insight into the intensity distribution of the overall diffraction pattern of the rotating slit.

To obtain such an insight, we must bear in mind that, as the slit is rotated about the center of the entrance pupil plane, every space-fixed point (v_o, w_o) of an arbitrary observation plane, u_o = constant, successively experiences the intensities of all points along a circle around the u-axis.

For convenience, the space-fixed v_o and w_o-axes may be placed into the tangential and meridional planes, respectively. The observation point may then be described by polar coordinates

$$v_o = \rho_o \sin \phi_o \quad \text{and} \quad w_o = \rho_o \cos \phi_o$$

If we now rotate the slit by the angle ϕ – and, consequently, the reference coordinate system by the same amount – we may determine the coordinates of the (space-fixed) observation point by the transformation

$$v = v_o \cos \phi + w_o \sin \phi = \rho_o \sin (\phi_o + \phi)$$

$$w = - v_o \sin \phi + w_o \cos \phi = \rho_o \cos (\phi_o + \phi)$$

The third optical coordinate, $u = u_o$, remains unchanged. These optical coordinates enter the intensity distribution function. Of course, in the new coordinate system, as in the preceding derivation, the geometrical image point is described by

$$X_1^* = R_1^* \sin \phi \quad \text{and} \quad Y_1^* = R_1^* \cos \phi$$

Thus, we obtain the instantaneous intensity distribution

$$i(u,v,w) = \left| \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left(jkCR_1^{*2}\right)^{\ell} \sum_{\lambda=0}^{2\ell} \binom{2\ell}{\lambda} \mu_o^{2\ell-\lambda} \nu_o^{\lambda} \right.$$

$$\left. \times (\sin\phi)^{2\ell-\lambda} (\cos\phi)^{\lambda} \overline{V_{\ell}^{2\ell-\lambda}}\left(u, \rho_o \sin(\phi_o + \phi)\right) \overline{W_{\ell}^{\lambda}}\left(u, \rho_o \cos(\phi_o+\phi)\right) \right|^2$$

where

$$\overline{V_{\ell}^{2\ell-\lambda}}\left(u, \rho_o \sin(\phi_o+\phi)\right) = \frac{\pi}{2} \sum_{s=0}^{\infty} (-j)^s (2s+1) \frac{J_{s+1/2}\left(\frac{u}{2}\mu_o^2\right)}{\left(\frac{u}{2}\mu_o^2\right)^{1/2}} G_s^{2\ell-\lambda}\left(\mu_o \rho_o \sin(\phi_o+\phi)\right)$$

$$\overline{W_{\ell}^{\lambda}}\left(u, \rho_o \cos(\phi_o+\phi)\right) = \frac{\pi}{2} \sum_{t=0}^{\infty} (-j)^t (2t+1) \frac{J_{t+1/2}\left(\frac{u}{2}\nu_o^2\right)}{\left(\frac{u}{2}\nu_o^2\right)^{1/2}} G_t^{\lambda}\left(\nu_o \rho_o \cos(\phi_o+\phi)\right)$$

$$G_s^{2\ell-\lambda}\left(\mu_o\rho_o \sin(\phi_o+\phi)\right) = \sum_{\sigma=0}^{\frac{2s+2\ell-\lambda-(1)}{2}} (-1)^{s-\sigma} \frac{N_{s,\sigma}^{2\ell-\lambda}}{D_s^{2\ell-\lambda}} \frac{J_{2s+2\ell-\lambda-2\sigma+1/2}\left(\mu_o\rho_o \sin(\phi_o+\phi)\right)}{\left(\mu_o\rho_o\sin(\phi_o+\phi)\right)^{1/2}}$$

$$G_t^{\lambda}\left(\nu_o\rho_o\cos(\phi_o+\phi)\right) = \sum_{\tau=0}^{\frac{2t+\lambda-(1)}{2}} (-1)^{t-\tau} \frac{N_{t,\tau}^{\lambda}}{D_t^{\lambda}} \frac{J_{2t+\lambda-2\tau+1/2}\left(\nu_o\rho_o\cos(\phi_o+\phi)\right)}{\left(\nu_o\rho_o\cos(\phi_o+\phi)\right)^{1/2}}$$

Every integrating device (as, for example, a photographic plate) accumulates instantaneous intensities, i.e., throughout half a slit revolution, it measures the intensity

$$i(u,v,w)_{tot} = \int_0^{\pi} i(u,v,w)\, d\phi$$

The integration involved is difficult to perform, even for the Gaussian image plane, $u = 0$. We expect, however, that the accumulated intensity at the geometrical image point, $i(0,0,0)_{tot}$, of the continuously rotating slit has a larger predominance over the accumulated intensity at a point in the surrounding field, $i(u,v,w)_{tot}$, than has the intensity at the Gaussian image point, $i(0,0,0)$, over that at the field point, $i(u,v,w)$, when the rectangular aperture is stationary. Rather than perform the integrations for a variety of specific field points, we prefer to rely on experiment to reveal the expected predominance. Knowledge of this predominance will indicate one method of improving the discrimination capability of an optical system for off-axis objects by appropriate diffraction-pattern manipulation.

Section 4
REFERENCES

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